PSEUDO-DIFFERENTIAL OPERATORS AND INTEGRABLE MODELS

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The importance of the theory of pseudo-differential operators in the study of non linear integrable systems is point out. Principally, the algebra Ξ of nonlinear (local and nonlocal) differential operators, acting on the ring of analytic functions $u_s(x, t)$, is studied. It is shown in particular that this space splits into several classes of subalgebras $\Sigma_{jr}, j = 0, \pm 1, r = \pm 1$ completely specified by the quantum numbers: s and (p, q) describing respectively the conformal weight (or spin) and the lowest and highest degrees. The algebra Σ_{++} (and its dual Σ_{--}) of local (pure nonlocal) differential operators is important in the sense that it gives rise to the explicit form of the second hamiltonian structure of the KdV system and that we call also the Gelfand-Dickey Poisson bracket. This is explicitly done in several previous studies, see for the moment [4, 5, 14]. Some results concerning the KdV and Boussinesq hierarchies are derived explicitly.

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1   Introduction

It's well known that integrable models [1], a subject attracting much more attention for several years both in physics and mathematics, are non linear hamiltonian systems with an appropriately large number of conserved quantities in involution making the system soluble.

In parallel to integrable models, another subject much more rich in structure, deal with $2d$ conformal field theories (CFT) [2] showing to be relevant in statistical mechanics and string theory [3]. The relation between these two classes of theories, namely the integrable models and the CFT's would be quite interesting and useful. For instance we recall that the KdV equation , a prototype of integrable models in two dimension can be obtained as a $Diff S^1$ flow if we identify the Hill's operator $L = \partial^2 + u_2$ with the space of quadratic differentials [4]. We know also that the second hamiltonian structure of the KdV system is nothing but the classical form of the Virasoro algebra or conformal symmetry [5]. In this context, we know that the KdV integrable model

$$\frac{\partial}{\partial t}u(x) = u\frac{\partial}{\partial x}u + \frac{\partial^3 u}{\partial x^3} \tag{1.1}$$

is a bi-Hamiltonian system in the sense that we can associate to the KdV equation two kind of Poisson brackets called the first and the second Hamiltonian structures. We have at equal times

$$\{u(x), u(y)\}_1 = \frac{\partial}{\partial x}\delta(x - y) \tag{1.2}$$

and

$$\{u(x), u(y)\}_2 = (u(x) + u(y))\frac{\partial}{\partial x}\delta(x - y) + \frac{1}{2}\frac{\partial^3}{\partial x^3}\delta(x - y) \tag{1.3}$$

The bracket $\{u(x), u(y)\}_2$ is nothing but the form reproducing the classical Virasoro algebra of the spin two algebra. On the other hand, using (1.1) with

$$H_3 = \int dx(\frac{1}{3!}u^3 - \frac{1}{2}(\frac{\partial u}{\partial x})^2) \tag{1.4}$$

then it's easily verified that

$$\frac{\partial u}{\partial t} = \{u(x), H_3\}_1 = u\frac{\partial}{\partial x}u + \frac{\partial^3 u}{\partial x^3} \tag{1.5}$$

showing that the KdV equation is Hamiltonian, see [6].

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In the language of $2d$ conformal field theory, the above mentioned currents u_s are taken in general as primary w_s satisfying the OPE

$$T(z)w_s(\omega) = \frac{s}{(z-\omega)^2}w_s(\omega) + \frac{w'_s(\omega)}{(z-\omega)},\tag{1.6}$$

or equivalently,

$$w_s = J^s . \tilde{w}_s\tag{1.7}$$

under a general change of coordinate (diffeomorphism) $x \rightarrow \tilde{x}(x)$ with $J = \frac{\partial \tilde{x}}{\partial x}$ is the associated Jacobian.

These w-symmetries [7, 8, 9] exhibit among other a non linear structure and are not Lie algebra in the standard way as they incorporate composite fields in their OPE.

In integrable models, and as signaled previously, these higher spin symmetries appear such that the Virasoro algebra W_2 defines the second Hamiltonian structure for the KdV hierarchy, W_3 for the Boussinesq and $W_{1+\infty}$ for the KP hierarchy and so one. These correspondences are achieved naturally in terms of pseudo-differential Lax operators.

$$\mathcal{L}_n = \sum_{j \in Z} u_{n-j}\partial^j,\tag{1.8}$$

allowing both positive as well as nonlocal powers of the differential ∂^j. The fields u_j of arbitrary conformal spin j did not define a primary basis. The construction of primary fields from the u_j one's is originated from the well known covariantization method of Di-Francesco -Itzykson-Zuber (DIZ)[10] showing that the primary W_j fields are given by adequate polynomials of u_j and their k-th derivatives $u_j^{(k)}$.

2 Basics notions and convention notations

2.1 The algebra of currents $u_s(x,t)$

We present in this first subsection the general setting of the basic properties of the algebra of currents, that we usually denote as $u_s(x,t)$. These are mathematical objects, of physical meaning, characterized by a quantum number namely the conformal weight (spin) $s > 1$.

This particular algebra is considered to be a semi-infinite dimensional space of huge infinite tensor algebra of arbitrary integer spin fields.

The currents $u_s(x,t)$ are playing a crucial role in physics through the conformal symmetry and its higher spin extensions called $w-$symmetry signaled in the introduction. These extended symmetries deal with analytic fields obeying a nonlinear closed algebra. They appear also in the study of higher differential operators involved in the analysis of nonlinear integrable models to be considered in the forthcoming parts of this work. We shall start however by defining our convention notations.

The two dimensional Euclidean space $\mathbb{R}^2 \cong \mathbb{C}$ is parametrized by the complex coordinates $z = t + ix$ and $\bar{z} = t - ix$. As a matter of convention, we set $z = z^+$ and $z = z^-$ so that the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$ are, respectively, represented by $\partial_+ = \partial$ and $\partial_- = \bar{\partial}$. The $so(2)$ Lorentz representation fields are described by one component tensors of the form $\psi_k(z, \bar{z})$ with $2k \in \mathbb{Z}$. $\mathbb{Z}$ is the set of relative integers. In two dimensional conformal field theories (CFT)[1], an interesting class of fields is given by the set of analytic fields $\phi_k(z)$. These are $SO(2) \cong U(1)$ tensor fields that obey the analyticity condition $\partial_- \phi_k(z) = 0$. In this case the conformal spin k coincides with the conformal dimension Δ. Note that under a $U(1)$ global transformation of parameter θ, the object $z^\pm$, $\partial_\pm$ and $\phi_k(z)$ transform as

$$z^{\pm\prime} = e^{\mp i\theta} z^\pm, \quad \partial'_\pm = e^{\pm i\theta} \partial_\pm, \quad \phi'_k(z) = e^{ik\theta} \phi_k(z) \tag{2.1}$$

so that $dz\partial_z$ and $(dz)^k \phi_k(z)$ remain invariant. In a pure bosonic theory, which is the purpose of the present study, only integer values of conformal spin k are involved. We denote by $\Xi^{(0,0)}$ the tensor algebra of analytic fields of arbitrary conformal spin. This is a completely reducible infinite dimensional $SO(2)$ Lorentz representation (module) that can be written as

$$\Xi^{(0,0)} = \bigoplus_{k \in \mathbb{Z}} \Xi_k^{(0,0)} \tag{2.2}$$

where the $\Xi_k^{(0,0)}$'s are one dimensional $SO(2)$ spin k irreducible modules. The upper indices $(0,0)$ carried by the spaces figuring in Eq. (2.2) are special values of general indices (p,q) to be introduced later on. The generators of these spaces are given by the spin k analytic fields $u_k(z)$. They may be viewed as analytic maps u_k which associate to each point z, on the unit circle S^1, the fields $u_k(z)$. For $k \geq 2$, these u_k fields can be thought of as the higher spin currents involved in the construction of the higher spin conformal currents or

w-algebras. As an example, the following fields:

$$w_2 = u_2(z), \quad w_3 = u_3(z) - \frac{1}{2}\partial_z u_2(z) \tag{2.3}$$

are the well-known spin-2 and spin-3 conserved currents of the Zamolodchikov w_3-algebra [7, 8, 9]. As in infinite dimensional spaces, elements Φ of the spin tensor algebra $\Xi^{(0,0)}$ in Eq. (2.2) are built from the vector basis $\{u_k, k \in \mathbb{Z}\}$ as follows:

$$\Phi = \sum_{k \in \mathbb{Z}} c(k) u_k \tag{2.4}$$

where only a finite number of the decomposition coefficients $c(k)$ is nonvanishing. Introducing the following scalar product $\langle , \rangle$ in the tensor algebra $\Xi^{(0,0)}$

$$\langle u_l, u_k \rangle = \delta_{k+l,1} \int dz \, u_{1-k}(z) u_k(z) \tag{2.5}$$

it is not difficult to see that the one dimensional subspaces $\Xi_k^{(0,0)}$ and $\Xi_{1-k}^{(0,0)}$ are dual to each other. As a consequence the tensor algebra $\Xi^{(0,0)}$ splits into two semi-infinite tensor subalgebras $\Sigma_+^{(0,0)}$ and $\Sigma_-^{(0,0)}$, respectively, characterized by positive and negative conformal spins as shown here below

$$\Sigma_+^{(0,0)} = \bigoplus_{k \succ 0} \Xi_k^{(0,0)} \tag{2.6}$$

$$\Sigma_-^{(0,0)} = \bigoplus_{k \succ 0} \Xi_{1-k}^{(0,0)} \tag{2.7}$$

From these equations we read in particular that $\Xi_0^{(0,0)}$ is the dual of $\Xi_1^{(0,0)}$ and if half integers were allowed, $\Xi_{1/2}^{(0,0)}$ would be self dual with respect to the form (2.5). Note that the product (2.5) carries a conformal spin structure since from dimensional arguments, it behaves as a conformal object of weight $\Delta = -1$

$$\Delta[\langle u_{1-k}, u_k \rangle] = -1 + (1-k) + k = 0 \tag{2.8}$$

Later on, we shall introduce a combined product $\langle\langle , \rangle\rangle$ built out of Eq. (2.5) and a pairing product $(,)$, see Eq. (3.34), of conformal weight $\Delta = 1$ so that we get

$$\Delta\left[\langle\langle , \rangle\rangle\right] = -1 + 1 = 0 \tag{2.9}$$

Moreover, the infinite tensor algebra $\Sigma_+^{(0,0)}$ of Eq. (2.6) contains, in addition to the spin-l current, all the W_n currents $n \geq 2$. These fields are used in the construction of higher spin local differential operators as it will be shown later on. Analytic fields with negative conformal spins [Eq. (2.7)] are involved in the building of nonlocal pseudodifferential operators. Both these local and nonlocal operators are needed in the derivation of the classical w_n-algebras from the Gelfand-Dickey algebra of $sl(n)$.

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2.2 Introducing pseudo-operators

Before going into more details in the next parts of this work, let's first start by a brief account of the basic properties of the space of higher order differential Lax operators. We have to fix that every pseudo-differential operator is completely specified by a conformal spin s, $s \in \mathbb{Z}$, two integers p and $q = p + n$, $n \geq 0$ defining the lowest and the highest degrees respectively and finally $(1 + q - p) = n + 1$ analytic fields $u_j(z)$ see for the moment [14]

We denote by $\mathcal{A}$ the huge algebra of all local and non local differential operators of arbitrary conformal spins and arbitrary degrees. One may expand $\mathcal{A}$ as

$$\mathcal{A} = \underset{p \leq q}{\oplus} \mathcal{A}^{(p,q)} = \underset{p \leq q}{\oplus} \ \underset{s \in \mathbb{Z}}{\oplus} \mathcal{A}_s^{(p,q)}, \ p, q, s \in \mathbb{Z} \tag{2.10}$$

where we have denoted by (p, q) the lowest and thee highest degrees respectively and by s the conformal spin. The vector space $\mathcal{A}^{(p,q)}$ of differential operators with given degrees (p, q) but undefined spin exhibits a Lie algebra structure with respect to the Lie bracket for $p \leq q \leq 1$. To see this, let us consider the set $\mathcal{A}_s^{(p,q)}$ of differential operators type

$$d_s^{(p,q)} := \sum_{i=p}^{q} u_{s-i}(z) \partial^i \tag{2.11}$$

It's straightforward to check that the commutator of two operators $d_s^{(p,q)}$ is an operator of conformal spin $2s$ and degrees $(p, 2q - 1)$. Since the Lie bracket $[.,.]$ acts as

$$[.,.] : \mathcal{A}_s^{(p,q)} \times \mathcal{A}_s^{(p,q)} \longrightarrow \mathcal{A}_{2s}^{(p,2q-1)} \tag{2.12}$$

Imposing the closure, one gets strong constraints on the spin s and the degrees parameters (p,q) namely

$$s = 0 \quad \text{and} \quad p \leq q \leq 1 \tag{2.13}$$

From these equations we learn in particular that the spaces $\mathcal{A}_0^{(p,q)}, p \leq q \leq 1$ admit Lie algebra structure with respect to the bracket Eq(2.13) provided that the Jacobi identity is fulfilled. This can be ensured by showing that the Leibnitz product is associative. Indeed given three arbitrary differential operators $d_{m_1}^{(p_1,q_1)}, d_{m_2}^{(p_2,q_2)}$ and $d_{m_3}^{(p_3,q_3)}$ we find that associativity follows by help of the identity

$$\sum_{l=0}^{i} \binom{i}{l} \binom{j}{k-l} = \binom{i+j}{k}, \tag{2.14}$$

where $\binom{i}{j}$ is the usual binomial coefficient.

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2.3 Some non linear differential equations

The KP equation

$$\partial_x(\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu') = \frac{3}{4}\frac{\partial^2 u}{\partial y^2}. \tag{2.15}$$

The non linear schrodinger equation [11, 12]

$$i\partial_t q = -q_{xx} + 2k(q^*q)q$$

$$i\partial_t q^* = -q^*_{xx} - 2k(q^*q)q^*, \tag{21}$$

where k is an arbitrary parameter measuring the strength of the non linear interaction and can be set to unity through a resealing of the dynamical variables q and q^*.

The two bosons Hierarchy [13]:

$$\partial_t u = (2h + u^2 - u_x)_x,$$

$$\partial_t h = (2uh + h_x)_x \tag{21}$$

This hierarchy is known to yields the non linear schrodinger equation once we set $u = -\frac{q_x}{q}$ and $h = -q^*q$.

The Liouville equation:

$$\bar{\partial}\partial\phi = exp(2\phi) \tag{2.16}$$

where ϕ is a Liouville scalar field. This is a conformally invariant field theory shown to be intimately related to the KdV equation thought the Virasoro symmetry.

2.4 Some pseudo differential operators

The algebra of pseudo-differential operators play fundamental role for the construction of the Kadomtsev-Petviashvili (KP) hierarchy in the Lax formalism. For the standard KP hierarchy, the associated pseudo-differential operator can be regarded as an ordinary integral operator, which enjoys the generalized Leibniz rule. We define the Lax operator of the KP hierarchy by

$$\mathcal{L}_{KP} = \partial + \sum_{j=1}^{\infty} u_{j+1}\partial^{-j}, \tag{2.17}$$

where u_j's are dependent variables of space x and time variables being introduced below. The coefficient of ∂^0 can be set zero without loss of generality. We assign the degree of

the differential operator ∂ one, standing for $\deg[\partial] = 1$, and assume that all the terms in the Lax operator have equal degree, *i.e.*, $\deg[u_j] = j$. Hence the algebra wrapping this operator is given by a quotient

$$\mathcal{L}_{KP} \in \mathcal{A}_1^{(-\infty,1)}/\mathcal{A}_1^{(0,0)} \tag{2.18}$$

with $\mathcal{A}_1^{(0,0)}$ (subalgebra of algebra $\mathcal{A}_1^{(-\infty,1)}$) the ring of the functions of spin 1. The second class of the important pseudo differential operators is given by the Burgers operators

$$\mathcal{L}_{Burgers} = \partial_x + u_1(x,t) \tag{2.19}$$

which is an element of the algebra

$$\mathcal{A}_1^{(0,1)} \equiv \mathcal{A}_1^{(-\infty,1)}/\mathcal{A}_1^{(-\infty,-1)}. \tag{2.20}$$

Another class of the important pseudo differential operators is given by the Korteweg-de Vries (KdV) operators

$$\mathcal{L}_{KdV} = \partial_x^2 + u_2(x,t) \tag{2.21}$$

the algebraic structure of this operator is given by the quotient

$$\mathcal{A}_2^{(0,2)}/\mathcal{A}_2^{(1,1)} \tag{2.22}$$

this operator can be seen as the second reduction of the KP hierarchy, indeed

$$\mathcal{L}_{KdV} = \left(\mathcal{L}_{KP}^2\right)_{\geq 0} = \left(\mathcal{L}_{KP}^2\right)_{+} \tag{2.23}$$

$\mathcal{L}_{KdV}$ is the part with only derivatives, *i.e.*, non-negative power terms in ∂. The same formalism applies for the Boussinesq operator which has the following form

$$L_{Bous\,sin\,esq} = \partial_x^3 + u_2(x,t)\partial_x + u_3(x,t) \tag{2.24}$$

which is an element of the following algebra

$$\mathcal{A}_3^{(0,3)}/\mathcal{A}_3^{(2,2)} \tag{2.25}$$

$\mathcal{L}_{Bous\,sin\,esq}$ can be seen as the third is the reduction of the KP hierarchy

$$\mathcal{L}_{Bous\,sin\,eq} = \left(\mathcal{L}_{KP}^3\right)_{\geq 0} = \left(\mathcal{L}_{KP}^3\right)_{+} \tag{2.26}$$

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3 Pseudo-differential operator's theory [14]

Here we describe, in a little bit more details, the basic features of nonlinear pseudo-differential operators on the ring of analytic functions. As quoted before, we show in particular that any such differential operator is completely specified by a weight (spin) $m, m \in \mathbb{Z}$, two integers p and $q = p + n, n \geq 0$ defining the lowest and highest degrees, respectively, and finally $(1 + q - p) = n + 1$ analytic fields $u_j(z)$. We also show that the set Ξ of all nonlinear differential operators admits a Lie algebra structure with respect to the commutator of differential operators built out of the Leibnitz product. Moreover we find that Ξ splits into $3 \times 2 = 6$ subalgebras Σ_{j+} and Σ_{j-}, $j = 0, \pm 1$ related to each other by two types of conjugations, namely, the spin and degree conjugations. The algebras Σ_{++} and Σ_{--} are of particular interest in this study as they are used in the construction of the Hamiltonian structure of nonlinear integrable models.

To that purpose we shall proceed as follows: First we introduce the space of differential operators of fixed spin m and fixed degrees (p, q). This space is referred hereafter to as $\Xi_m^{(p,q)}$. Then we consider the set $\Xi^{(p,q)}$ of nonlinear operators of fixed degrees (p, q) but arbitrary spin. Finally we build the desired space.

3.1 The $\Xi_m^{(p,q)}$ space

To begin, we remark that $\Xi_m^{(p,q)}$ is the space of differential operators whose elements $d_m^{(p,q)}(u)$ are the generalization of the well-known scalar Lax operator involved in the analysis of the so-called KdV hierarchies and in Toda theories [15, 16, 17]. The simplest example is given by the Hill operator

$$L = \partial^2 + u(z) \tag{3.1}$$

which plays an important role in the study of the Liouville theory and in the KdV equation. A natural generalization of the above relation is given by

$$d_m^{(p,q)}(u) = \sum_{i=p}^{q} u_{m-i}(z)\partial^i \tag{3.2}$$

where the $u_{m-i}(z)$'s are analytic fields of spin $(m - i)$, p and q, with $p \geq q$, are integers that we suppose are positive for the moment. We shall refer hereafter to p as the lowest degree of $d_m^{(p,q)}(u)$ and q as the highest one. We combine these two features of Eq. (3.2) by setting

$$Deg(d_m^{(p,q)}(u)) = (p, q) \tag{3.3}$$

m is the conformal spin of the $(1 + q - p)$ monomes of the right hand side (rhs) of Eq. (3.2) and then of $d_m^{(p,q)}(u)$ itself. As for the above relation, we set

$$\Delta(d_m^{(p,q)}(u)) = m \tag{3.4}$$

Putting $m = 2, p = 0$, and $q = 2$ together with the special choices $u_0(z) = 1$ and $u_1(z) = 0$ in Eq. (3.2), we recover Eq. (3.1) as a particular object. Moreover, Eq. (3.2) which is well defined for $q \geq p \geq 0$ may be extended to negative integers by introducing pseudodifferential operators of the type ∂^{-k}, $k \geq 0$, whose action on the fields $u_s(z)$ is defined as

$$\partial^{-k} u_s(z) = \sum_{l=0}^{\infty} (-1)^l C_{k+l-1}^l u_s^{(l)}(z) \partial^{-k-l} \tag{3.5}$$

where $u_s^{(l)}(z)$ is the l−th derivative of $u_s(z)$. As can be checked by using the Leibnitz rule, Eq. (3.5) obeys the expected property

$$\partial^k \partial^{-k} u_s(z) = u_s(z) \tag{3.6}$$

A natural representation basis of nonlinear pseudodifferential operators of spin m and negative degrees (p, q) is given by

$$\delta_m^{(p,q)}(u) = \sum_{i=p}^{q} u_{m-i}(z) \partial^i \tag{3.7}$$

This configuration, which is a direct extension of Eq. (3.2), is useful in the study of the algebraic structure of the spaces $\Xi_m^{(p,q)}$ and $\Xi^{(p,q)}$. Note by the way that we can use another representation of pseudodifferential operators, namely, the Volterra representation. The latter is convenient in the derivation of the second Hamiltonian structure of higher conformal spin integrable theories. Note also that Eq. (3.5) is a special pseudodifferential operator of the type of Eq. (3.7) with $m = s - k, p = -\infty$ and $q = -k$.

Using Eqs. (3.2) and (3.7), one sees that operators with negative lowest degrees p and positive highest degrees q denoted by $D_m^{(p,q)}[u]$ split as

$$D_m^{(p,q)}[u] = \delta_m^{(p,q)}(u) + d_m^{(p,q)}(u) \tag{3.8}$$

More generally we have

$$D_m^{(p,q)}[u] = D_m^{(p,k)}(u) + D_m^{(k+1,q)}(u) \tag{3.9}$$

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for any integers $p \leq k \leq q$. As a consequence, one finds that the operation (3.3) obeys the rule

$$(p, q) = (p, k) + (k + 1, q) \tag{3.10}$$

for any three integers such that $p \leq k < q$. Now let $\Xi_m^{(p,q)}$; m, p, and q integers with $q \geq p$, be the set of spin m differential operators of degrees (p, q). With respect to the usual addition and multiplication by C numbers, $\Xi_m^{(p,q)}$ behaves as $(1 + q - p)$ dimensional space generated by the vector basis

$$\left\{ D_m^{(p,q)} [u] = \sum_{i=p}^{q} u_{m-i} \partial^i; \quad p \leq i \leq q \right\}$$

Thus the space decomposition of $\Xi_m^{(p,q)}$ reads as

$$\Xi_m^{(p,q)} = \bigoplus_{i=p}^{q} \Xi_m^{(i,i)} \tag{3.11}$$

where the $\Xi_m^{(i,i)}$'s are one dimensional spaces given by

$$\Xi_m^{(i,i)} = \Xi_{m-i}^{(0,0)} \otimes \partial^i \tag{3.12}$$

Setting $i = 0$, one discovers the space $\Xi_m^{(0,0)}$ defined previously. Remark that the number of independent fields $u_j(z)$ involved in Eqs. (3.2) and (3.7)-(3.9) is equal to the dimension of $\Xi_m^{(p,q)}$. In the next subsection we shall show that among all the spaces $\Xi_m^{(p,q)}$, m, p and q arbitrary integers, only the sets $\Xi_0^{(p,q)}$ with $p < q < 1$ admit a Lie algebra structure with respect to the bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ constructed out of the Leibnitz product $\circ$. Because of the relations (3.9)-(3.10) these spaces obey

$$\Xi_0^{(p,k)} \subset \Xi_0^{(p,q)}, \quad \Xi_0^{(k+1,q)} \subset \Xi_0^{(p,q)} \tag{3.13}$$

and are then subalgebras of the maximal Lie algebra $\Xi_0^{(-\infty, 1)}$ of Lorentz scalar differential operators of highest degree less than 2.

3.2 Explicit formulas

Performing computations, we find the following Leibnitz rules:

$$\begin{aligned}
\partial u &= u\partial + u', \\
\partial^2 u &= u\partial^2 + 2u'\partial + u'', \\
\partial^3 u &= u\partial^3 + 3u'\partial^2 + 3u''\partial + u''',
\end{aligned} \tag{21}$$

and

$$\partial^{-1}u = u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - u'''\partial^{-4} + ...,$$

$$\partial^{-2}u = u\partial^{-2} - 2u'\partial^{-3} + 3u''\partial^{-4} - 4(\tfrac{1}{2})^3 u'''\partial^{-5} + ..., \tag{22}$$

$$\partial^{-3}u = u\partial^{-3} - 3\tfrac{1}{2}u'\partial^{-4} + 6(\tfrac{1}{2})^2 u''\partial^{-5} - 10(\tfrac{1}{2})^3 u'''\partial^{-6} + ...$$

The general expressions are given, for positive value of n, by the Leibnitz rules

$$\partial^n u = \sum_{i=0}^{n} c_n^i u^{(i)} \partial^{n-i},$$

$$\partial^{-n}u = \sum_{i=0}^{\infty} (-1)^i c_{n+i-1}^i u^{(i)} \partial^{-n-i}. \tag{23}$$

3.3 The $\Xi_0^{(p,q)}$ Lie algebra

We start by considering differential operators of the type of Eq.(3.2). The Leibnitz product 0 acts on $\Xi_m^{(p,q)}$; $q \geq p \geq 0$ as

$$\circ : \Xi_m^{(p,q)} \times \Xi_m^{(p,q)} \longrightarrow \Xi_{2m}^{(p,2q)} \tag{3.14}$$

or equivalently

$$d_m^{(p,q)}(u) \times d_m^{(p,q)}(v) = \sum_{k=p}^{2q} w_{2m-k}\partial^k = d_{2m}^{(p,2q)}(w) \tag{3.15}$$

where the composite analytic fields w_{2m-k} are given by

$$w_{2m-k}(z) = \sum_{i=p}^{q}\sum_{j=p}^{q} C_i^{k-j} u_{m-i} v_{m-j}^{(i+j-k)}, \quad C_r^s = \frac{r!}{s!\,(r-s!)}, \quad 0 \leq s \leq r. \tag{3.16}$$

Similar relations may be written down for the commutator of two differential operators. We have

$$[,] : \Xi_m^{(p,q)} \times \Xi_m^{(p,q)} \longrightarrow \Xi_{2m}^{(p,2q-1)} \tag{3.17}$$

showing that in general the space $\Xi_m^{(p,q)}$ is not closed under the action of the bracket $[,]$. Imposing the closure, one gets strong constraints on the integers m, p, and q, namely,

$$m = 0, \quad 0 \leq p \leq q \leq 1 \tag{3.18}$$

The spaces $\Xi_0^{(p,q)}$ satisfying the above constraint equations then exhibit a Lie algebra structure provided that the Jacobi identity is fulfilled. This can be ensured by showing that the Leibnitz product is associative. Indeed given three arbitrary differential operators

$$D_1 = d_{m_1}^{(p_1,q_1)}, \quad D_2 = d_{m_2}^{(p_2,q_2)}, \quad D_3 = d_{m_3}^{(p_3,q_3)} \tag{3.19}$$

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we find that associativity

$$D_1 \circ (D_2 \circ D_3) = (D_1 \circ D_2) \circ D_3 \tag{3.20}$$

follows by with help of the identity

$$\sum_{l=0}^{i} C_i^l C_j^{k-l} = C_{i+j}^k \tag{3.21}$$

Note that the constraint relations (3.18) give rise to three Lie algebras $\Xi_0^{(0,0)}$, $\Xi_0^{(1,1)}$, and $\Xi_0^{(0,1)}$ obeying the obvious space decomposition

$$\Xi_0^{(0,0)} \oplus \Xi_0^{(1,1)} = \Xi_0^{(0,1)} \tag{3.22}$$

Here, one recognizes $\Xi_0^{(1,1)}$ as just the Lie algebra of vector fields on the circle, namely, $Diff(S^1)$. $\Xi_0^{(0,0)}$ is the one dimensional trivial $\Xi_0^{(0,1)}$ ideal as that in Eq. (3.22). The above analysis can be extended to the pseudodifferential operators $\delta_m^{(p,q)}(u)$; $p \leq q < 0$. Using analogous calculations, we find that the spaces $\Xi_m^{(p,q)}$ are closed with respect to the commutator $[,]$ provided that

$$m = 0, \quad (p+1)/2 \leq q \leq -1 \tag{3.23}$$

Here also associativity ensures that the vector spaces $\Xi_0^{(p,q)}$ satisfying Eqs. (3.23) admit Lie algebra structures. These spaces as well as $\Xi_0^{(0,1)}$ are in fact subalgebras of the huge Lie algebra $\Xi_0^{(-\infty,1)}$. The latter has the remarkable space decomposition

$$\Xi_0^{(-\infty,1)} = \Xi_0^{(-\infty,-1)} \oplus \Xi_0^{(0,1)} \tag{3.24}$$

where $\Xi_0^{(-\infty,1)}$ is the Lie algebra of Lorentz scalar pure pseudodifferential operators of higher degree $q = -1$. Note finally that for a given $k \leq 1$, we have

$$\Xi_0^{(-\infty,k-1)} \subset \Xi_0^{(-\infty,k)} \tag{3.25}$$

and by Eq. (3.17)

$$\left[\Xi_0^{(-\infty,l)}, \Xi_0^{(-\infty,k)} \right] \subset \Xi_0^{(-\infty,k+l-1)} \tag{3.26}$$

which in turn shows that all $\Xi_0^{(-\infty,k-n)}, n > 0$ are ideals of $\Xi_0^{(-\infty,k)}$. Thus the spaces $\Xi_0^{(p,q)}$, $p \leq q < 1$, may be viewed as coset algebras

$$\Xi_0^{(p,q)} = \Xi_0^{(-\infty,q)} / \Xi_0^{(-\infty,p-1)} \tag{3.27}$$

Note that the space of Lorentz scalar differential operators $\Xi_0^{(p,q)}$ emerges as a special set of the algebra of all nonlinear differential operators.

3.4 The $\Xi^{(p,q)}$ algebra $(p \leq q \leq 1)$

So far we have seen that the closure of the commutator of higher differential operators imposes constraints on the conformal spin and on the degrees, Eqs. (3.18) and (3.23). The restriction on the spin can be overcome by using the spin tensor algebra $\Xi^{(0,0)}$ given by Eq.(2.2) instead of $\Xi_0^{(0,0)}$. As a consequence, we get a larger set of differential operators than $\Xi_m^{(p,q)}$. This space to which we refer hereafter to as $\Xi^{(p,q)}$, $q \geq p$, is constructed as follows:

$$\Xi^{(p,q)} = \bigoplus_{m \in \mathbb{Z}} \Xi_m^{(p,q)}, \quad p, q \in \mathbb{Z} \tag{3.28}$$

The elements $D^{(p,q)}$ of this infinite dimensional space are differential operators with fixed degrees (p, q) but arbitrary spin. They read as

$$D^{(p,q)} = \sum_{m \in \mathbb{Z}} C(m) D_m^{(p,q)}, \tag{3.29}$$

where only a finite number of the $c(m)$'s are not vanishing. Setting $p = q = 0$, we get the tensor algebra $\Xi_0^{(0,0)}$. As for the spaces $\Xi_m^{(p,q)}$, the set $\Xi^{(p,q)}$ given by Eq. (3.28) exhibits a Lie algebra structure with respect to the bracket $[,]$ for p$\leq q \leq 1$. Thus analogous relations to Eqs. (3.35)-(3.27) can be written down for $\Xi^{(p,q)}$. In particular $\Xi^{(p,q)}, p \leq q \leq 1$, may be viewed as a coset algebra of $\Xi^{(-\infty,q)}$ by $\Xi^{(-\infty,p-1)}$, i.e.,

$$\Xi^{(p,q)} = \Xi^{(-\infty,q)} / \Xi^{(-\infty,p-1)}. \tag{3.30}$$

3.5 The algebra of differential operators Ξ

This is the algebra of differential operators of arbitrary spins and arbitrary degrees. It is obtained from Eq. (3.28) by summing over all the allowed degrees of the spaces $\Xi^{(p,q)}$. In some sense, it is the degree tensor algebra of $\Xi^{(p,q)}$

$$\Xi = \bigoplus_{p \leq q} \Xi^{(p,q)} \tag{3.31}$$

or equivalently

$$\Xi = \bigoplus_{p \in \mathbb{Z}} \left[\bigoplus_{n \in \mathbb{N}} \Xi^{(p,p+n)} \right] \tag{3.32}$$

Note that this infinite dimensional space is closed under the Leibnitz commutator without any constraint. Note also that Ξ is just the combined spin and degree tensor algebra since we have

$$\Xi = \bigoplus_{p \in \mathbb{Z}} \left[\bigoplus_{n \in \mathbb{N}} \left(\bigoplus_{m \in \mathbb{Z}} \Xi_m^{(p,p+n)} \right) \right] \tag{3.33}$$

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A remarkable property of Ξ is that it splits into six infinite subalgebras Σ_{q+} and Σ_{q-}, $q = 0, \pm 1$, related to each others by conjugation of the spin and degrees. Indeed given two integers $q \geq p$, it is not difficult to see that the spaces $\Xi^{(p,q)}$ and $\Xi^{(-1-q,-1-p)}$ are dual with respect to the pairing product $(,)$ defined as

$$\left(D^{(r,s)}, D^{(p,q)}\right) = \delta_{1+r+q,0}\delta_{1+s+p,0}res\left[D^{(r,s)} \circ D^{(p,q)}\right], \tag{3.34}$$

where the residue operation (res) is given by

$$res\left[\partial^i\right] := \delta_{i+1,0}. \tag{3.35}$$

As already shown in Eq. (2.9), note that the operation res carries a conformal weight $\Delta = 1$ and then the residue of any operator $D_m^{(p,q)}$ is

$$res\left(\sum_{i=p}^{q} u_{m-i}\left(z\right)\partial^i\right) = u_{m+1}\left(z\right) \tag{3.36}$$

if $p \leq -1$ and $q \geq -1$ and zero elsewhere. We have, for instance,

$$\begin{aligned}\Delta\left[res\left(\partial^{-1}\right)\right] &= 1 - 1 = 0 \\ \Delta\left[res\left(u_{m+1}\partial^{-1}\right)\right] &= 1 + (m+1) - 1 = m + 1\end{aligned} \tag{3.37}$$

Then using Eqs. (3.34)-(3.37), we can decompose Ξ as

$$\Xi = \Xi_- \oplus \Xi_+ \tag{3.38}$$

with

$$\Xi_+ = \bigoplus_{p\geq 0}\left[\bigoplus_{r\geq 0}\Xi^{(p,p+r)}\right], \tag{3.39}$$

$$\Xi_- = \bigoplus_{p\geq 0}\left[\bigoplus_{r\geq 0}\Xi^{(-1-p-r,-1-p)}\right]. \tag{3.40}$$

The + and - down stairs indices carried by Ξ_+ and Ξ_- refer to the positive and negative degrees, respectively. Knowing that the $\Xi^{(p,p+r)}$ spaces can also be decomposed as in Eqs. (2.6) and (2.7), with a slight modification due to Eqs. (2.8) (2.9), (3.37), and (3.38)

$$\Xi^{(p,p+r)} = \Sigma_-^{(p,p+r)} \oplus \Sigma_0^{(p,p+r)} \oplus \Sigma_+^{(p,p+r)}, \tag{3.41}$$

where $\Sigma_-^{(p,p+r)}$ and $\Sigma_+^{(p,p+r)}$ denote the spaces of differential operators of negative and positive definite spin. They reads as

$$\Sigma_-^{(p,p+r)} = \bigoplus_{m\succ 0}\Xi_{-m}^{(p,p+r)}, \tag{3.42}$$

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$$\Sigma_0^{(p,p+r)} = \Xi_0^{(p,p+r)}, \tag{3.43}$$

$$\Sigma_+^{(p,p+r)} = \bigoplus_{m \succ 0} \Xi_m^{(p,p+r)}, \tag{3.44}$$

$\Sigma_0^{(p,p+r)}$ is the space of Lorentz scalar differential operators. Then plugging in Eqs. (3.38)-(3.40), we find that z decomposes into $3 \times 2 = 6$ subalgebras as

$$\Xi = \bigoplus_{q=0,\pm 1} \left[\Sigma_{q+} \oplus \Sigma_{q-} \right], \tag{3.45}$$

where

$$\Sigma_{q+} = \bigoplus_{p \geq 0} \left[\bigoplus_{r \geq 0} \Sigma_q^{(p,p+r)} \right], \quad q = 0, \pm 1 \tag{3.46}$$

$$\Sigma_{q-} = \bigoplus_{p \geq 0} \left[\bigoplus_{r \geq 0} \Sigma_q^{(-1-p-r,-1-p)} \right]. \tag{3.47}$$

Introducing the combined scalar product $\langle\langle , \rangle\rangle$ built out of the product (2.5) and the pairing (3.34), namely,

$$\left\langle\left\langle D_m^{(r,s)}, D_m^{(p,q)} \right\rangle\right\rangle = \delta_{n+m,0}\delta_{1+r+q,0}\delta_{1+s+p,0} \int dz \; res \left[D_m^{(r,s)} \circ D_{-m}^{(-1-s,-1-r)} \right] \tag{3.48}$$

one sees that Σ_{++}, Σ_{0+} and Σ_{-+} are the duals of Σ_{--}, Σ_{0-} and Σ_{+-}, respectively. Note that Σ_{0-} is just the algebra of Lorentz scalar pseudo-operators of higher degree (-1) considered later on.

We conclude this section by making two comments: First we remark that Σ_{++} is the space of local differential operators of positive definite spins and positive degrees. Σ_{--} however, is the Lie algebra of nonlocal operators of negative definite spins and negative degrees. It is these two subalgebras that will be considered in the remainder of this study. The spaces Σ_{0+} and Σ_{0-} are very special subalgebras and will be analyzed in a future article.

The second comment we want to make is to note that once we know that local and nonlocal differential operators are completely specified by the spin and the degrees, one may reverse the previous discussion. Start with Ξ, the algebra of all possible nonlinear operators and decompose it with respect to subspaces with definite spin and definite degrees as

$$\Xi = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{k \in \mathbb{Z}} \bigoplus_{m \in \mathbb{Z}} \Xi_m^{(p,p+k)}. \tag{3.49}$$

So that the basic objects in Ξ are $\Xi_m^{(p,p+k)}$. All the sets introduced above appear here as special subspaces. Their conformal as well as the degree properties are summarized in the

following table:

$$
\begin{array}{|c|c|c|}
\hline
 & \text{Conformal weight } \Delta & \text{Degrees: } Deg \\
\hline
\Xi_m^{(p,q)} & m & (p,q) \\
\hline
\Xi^{(p,q)} & indefinite & (p,q) \\
\hline
\langle,\rangle_{\Xi_m^{(0,0)}} & -1 & (0,0) \\
\hline
res & 1 & (0,0) \\
\hline
\ll,\gg & 0 & (0,0) \\
\hline
\end{array}
\tag{3.50}
$$

4 sl_n KdV-hierarchy

The aim of this section is to present some results related to the KdV hierarchy. Using our convention notations and the analysis that we developed previously, we will perform hard algebraic computations and derive the KdV hierarchy.

These computations are very hard and difficult to realize in the general case. We will simplify this study by limiting our computations to the leading orders of the hierarchy namely the sl_2-KdV and sl_3-Boussinesq integrable hierarchies.

Our contribution to this study consists in extending known results by increasing the order of computations a fact which leads us to discover more important properties as we will explicitly show. As an original result, we will build the deformed sl_3-Boussinesq hierarchy and derive the associated flows. Some other important results are also presented.

4.1 sl_2-KdV hierarchy

Let's consider the sl_2 Lax operator

$$
\mathcal{L}_2 = \partial^2 + u_2 \tag{4.1}
$$

whose 2th root is given by

$$
\begin{aligned}
\mathcal{L}^{\frac{1}{2}} &= \Sigma_{i=-1} b_{i+1} \partial^{-i} \\
&= \Sigma_{i=-1} a_{i+1} \partial^{-i}
\end{aligned}
\tag{4.2}
$$

This 2th root of $\mathcal{L}_2$ is an object of conformal spin $[\mathcal{L}^{\frac{1}{2}}] = 1$ that plays a central role in the derivation of the Lax evolutions equations.

Performing lengthy but straightforward calculations we compute the coefficients b_{i+1}

of $\mathcal{L}^{\frac{1}{2}}$ up to $i = 7$ given by

$$b_0 = 1$$

$$b_1 = 0$$

$$b_2 = \tfrac{1}{2}u$$

$$b_3 = -\tfrac{1}{4}u'$$

$$b_4 = -\tfrac{1}{8}u^2 + \tfrac{1}{2}(\tfrac{1}{2})^2 u''$$

$$b_5 = -\tfrac{1}{2}(\tfrac{1}{2})^3 u''' + \tfrac{3}{8}uu'$$

$$b_6 = \tfrac{1}{16}u^3 - \tfrac{7}{4}(\tfrac{1}{2})^2 uu'' - \tfrac{11}{8}(\tfrac{1}{2})^2(u')^2 + \tfrac{1}{2}(\tfrac{1}{2})^4 u''''$$

$$b_7 = -\tfrac{15}{32}u^2u' + (\tfrac{1}{2})^3(\tfrac{15}{2}u''u' + \tfrac{15}{4}uu'''') - \tfrac{1}{2}(\tfrac{1}{2})^5 u^{(5)}$$

$$b_8 = -\tfrac{5}{128}u^4 + (\tfrac{1}{2})^2(\tfrac{55}{16}u''u^2 + \tfrac{85}{16}uu'^2) - (\tfrac{1}{2})^4(\tfrac{31}{4}uu'''' + \tfrac{91}{8}u''^2 + \tfrac{37}{2}u'u''') + \tfrac{1}{2}(\tfrac{1}{2})^6 u^{(6)}$$

$$(4.3)$$

and

$$b_9 = \tfrac{35}{64}u^3u' - \tfrac{175}{4}(\tfrac{1}{2})^3\left(uu'u'' + \tfrac{1}{4}u'^3 + \tfrac{1}{4}u^2u'''\right) + \tfrac{7}{4}(\tfrac{1}{2})^5\left(9uu^{(5)} + 25u^{(4)}u' + 35u'''u''\right)$$

$$\qquad - \tfrac{1}{2}(\tfrac{1}{2})^7 u^{(7)}$$

$$b_{10} = \tfrac{7}{256}u^5 - \tfrac{35}{32}(\tfrac{1}{2})^2\left(\tfrac{23}{2}u^2u'^2 + 5u^3u''\right) + \tfrac{7}{4}(\tfrac{1}{2})^4\left(\tfrac{73}{4}u^2u^{(4)} + \tfrac{227}{4}uu''^2 + \tfrac{337}{4}u''u'^2 + 89uu'u'''\right)$$

$$\qquad - \tfrac{3}{4}(\tfrac{1}{2})^6\left(\tfrac{631}{3}u''u^{(4)} + 233uu'''^2 + 135u'u^{(5)}\right) + \tfrac{1}{2}(\tfrac{1}{2})^8 u^{(8)}$$

$$(4.4)$$

These results are obtained by using the identification $\mathcal{L}_2 = \mathcal{L}^{\frac{1}{2}} \times \mathcal{L}^{\frac{1}{2}}$. Note that by virtue of Eq.(4.2), the coefficients a_{i+1} are shown to be functions of b_{i+1} and their derivatives in the following way

$$a_{i+1} = \sum_{s=0}^{i-1}(\frac{1}{2})^s c_{i-1}^s b_{i+1-s}^{(s)}, \qquad (4.5)$$

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We obtain the results:

$$a_0 = 1$$

$$a_2 = \tfrac{1}{2}u$$

$$a_4 = -\tfrac{1}{8}u^2$$

$$a_6 = \tfrac{1}{16}u^3 + \tfrac{1}{8}(\tfrac{1}{2})^2(u'^2 - 2uu'')$$

$$a_8 = -\tfrac{5}{128}u^4 + \tfrac{5}{8}(\tfrac{1}{2})^2\left(u^2u'' - \tfrac{1}{2}u'^2u\right) + \tfrac{1}{4}(\tfrac{1}{2})^4\left(u'''u' - uu^{(4)} - \tfrac{1}{2}u''^2\right)$$

$$a_{10} = \tfrac{7}{256}u^5 + \tfrac{35}{64}(\tfrac{1}{2})^2\left(\tfrac{1}{2}u^2u'^2 - u^3u''\right) + \tfrac{7}{4}(\tfrac{1}{2})^4\left(\tfrac{3}{4}u^{(4)}u^2 + \tfrac{7}{4}u''^2u - \tfrac{3}{4}u'^2u'' - uu'u'''\right)$$

$$\quad + \tfrac{1}{4}(\tfrac{1}{2})^6\left(u'u^{(5)} + \tfrac{1}{2}u'''^2 - uu^6\right)$$

$$a_{12} = -\tfrac{21}{1024}u^6 + \tfrac{105}{64}(\tfrac{1}{2})^2\left(u^4u'' - \tfrac{1}{2}u^3u'^2\right)$$

$$\quad + \tfrac{1}{16}(\tfrac{1}{2})^4\left(147uu''u'^2 + \tfrac{189}{2}u^2u'u''' - \tfrac{1029}{4}u^2u''^2 - 63u^3u^{(4)} - \tfrac{105}{8}u'^4\right)$$

$$\quad + \tfrac{1}{4}(\tfrac{1}{2})^6\left(16u''^3 + 9u^2u^{(6)} - 27u'u''u''' - \tfrac{45}{2}u'^2u^{(4)} - \tfrac{69}{4}u''^2u + \tfrac{153}{2}uu''u^{(4)} - \tfrac{27}{2}uu'u^{(5)}\right)$$

$$\quad + \tfrac{1}{4}(\tfrac{1}{2})^8\left(u'u^{(7)} + u'''u^{(5)} - u''u^{(6)} - uu^{(8)} - \tfrac{1}{2}u^{(4)2}\right)$$

$$\quad \vdots$$

$$(4.6)$$

with

$$a_{2k+1} = \sum_{s=0}^{2k-1}(\tfrac{1}{2})^s c_{2k-1}^s b_{2k+1-s}^{(s)} = 0, \qquad k = 0,1,2,3,... \tag{4.7}$$

Now having derived the explicit expression of $\mathcal{L}^{\frac{1}{2}}$, we are now in position to write the explicit forms of the set of sl_n KdV hierarchy. These equations defined as

$$\frac{\partial \mathcal{L}}{\partial t_k} = \{(\mathcal{L}^{\frac{k}{2}})_+, \mathcal{L}\}, \tag{4.8}$$

are computed up to the first three flows t_1, t_3, t_5. We work out these equations by adding

other flows namely t_7 and t_9. We find

$$u_{t_1} = u'$$

$$u_{t_3} = \tfrac{3}{2}uu' + (\tfrac{1}{2})^2 u'''$$

$$u_{t_5} = \tfrac{15}{8}u^2 u' + 5(\tfrac{1}{2})^2(u'u'' + \tfrac{1}{2}uu''') + (\tfrac{1}{2})^4 u^{(5)}$$

$$u_{t_7} = \tfrac{35}{16}u^3 u' + \tfrac{35}{8}(\tfrac{1}{2})^2(4uu'u'' + u'^3 + u^2 u''') + \tfrac{7}{2}(uu^{(5)} + 3u'u^{(4)} + 5u''u''')(\tfrac{1}{2})^4 + (\tfrac{1}{2})^6 u^{(7)}$$

$$\begin{aligned}
u_{t_9} = {} & 18(\tfrac{1}{2})^6 u'u^{(6)} + \tfrac{651}{8}(\tfrac{1}{2})^4 u'(u'')^2 + \tfrac{315}{128}u^4 u' + \tfrac{483}{8}(\tfrac{1}{2})^4 u'^2 u''' + \tfrac{315}{16}(\tfrac{1}{2})^2 uu'^3 + \tfrac{189}{4}(\tfrac{1}{2})^4 uu^{(4)}u' \\
& + \tfrac{315}{8}(\tfrac{1}{2})^2 u^2 u'u'' + \tfrac{315}{4}(\tfrac{1}{2})^4 uu'u''' + 63(\tfrac{1}{2})^6 u''u^{(4)} + \tfrac{105}{16}(\tfrac{1}{2})^2 u^3 u''' + 42(\tfrac{1}{2})^6 u^{(5)}u'' \\
& + \tfrac{63}{8}(\tfrac{1}{2})^4 u^2 u^{(5)} + (\tfrac{1}{2})^8 u^{(9)} + \tfrac{9}{2}(\tfrac{1}{2})^6 uu^{(7)}
\end{aligned}$$

$$(4.9)$$

Ssome important remarks are in order:

1. The flow parameters t_{2k+1} has the following conformal dimension $[\partial_{t_{2k+1}}] = -[t_{2k+1}] = 2k+1$ for $k = 0, 1, 2, ...,$.

2. A remarkable property of the sl_2 KdV hierarchy is about the degree of non linearity of the evolution equations Eq.(4.9). We present in the following table the behavior of the higher non-linear terms with respect to the first leading flows $t_1, ..., t_9$ and give the behavior of the general flow parameter t_{2k+1}.

Flows	The higher n.l. terms	Degree of n linearity
t_1	$u^0 u' = u'$	0
t_3	$\frac{3}{2} u u'$	1 (quadratic)
t_5	$\frac{15}{2^3} u^2 u'$	2 (cubic)
t_7	$\frac{35}{2^4} u^3 u'$	3
t_9	$\frac{315}{2^7} u^4 u'$	4
...	...	...
t_{2k+1}	$\eta(2k+1)(2k-1)u^k u'$	(k),

$$(4.10)$$

where η is an arbitrary constant.

This result shows among others that the evolution equations Eq.(4.9) exhibit at most a nonlinearity of degree (k) associated to a term proportional to $(2k+1)(2k-1)u^k u'$. The particular case $k = 0$ corresponds to linear wave equation.

3. The results show a correspondence between the flows t_{2k+1} and the coefficient number $(\frac{1}{2})^{2(k-s)}, 0 \leq s \leq k$. Particularly, the higher term $(\frac{1}{2})^{2(k)}$ is coupled to the $k-th$ prime derivative of u_2 namely $u^{(k)}$.

4. Once the non linear terms in the evolution equations are ignored, there will be no solitons in the KdV-hierarchy as the latter's are intimately related to non linearity.

4.2 sl_3-Boussinesq Hierarchy

The same analysis used in deriving the sl_2-KdV hierarchy is actually extended to build the sl_3-Boussinesq hierarchy. The latter is associated to the momentum Lax operator $\mathcal{L}_3 = \partial^3 + u_2 \partial + u_3$ whose $3 - th$ root reads as

$$
\begin{aligned}
\mathcal{L}^{\frac{1}{3}} &= \Sigma_{i=-1} b_{i+1} \partial^{-i} \\
&= \Sigma_{i=-1} a_{i+1} \partial^{-i}
\end{aligned}
$$

$$(4.11)$$

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in such way that $\mathcal{L}_3 = \mathcal{L}^{\frac{1}{3}}\mathcal{L}^{\frac{1}{3}}\mathcal{L}^{\frac{1}{3}}$. Explicit computations lead to

$$b_0 = 1$$

$$b_1 = 0$$

$$b_2 = \tfrac{1}{3}u_2$$

$$b_3 = \tfrac{1}{3}u_3 - \tfrac{2}{6}u_2'$$

$$b_4 = -\tfrac{1}{9}u_2^2 - \tfrac{2}{6}u_3' + \tfrac{8}{9}(\tfrac{1}{2})^2 u_2''$$

$$b_5 = -\tfrac{2}{9}u_2 u_3 + \tfrac{8}{18}u_2 u_2' + \tfrac{8}{9}(\tfrac{1}{2})^2 u_3'' - \tfrac{8}{9}(\tfrac{1}{2})^3 u_3'''$$

$$b_6 = \tfrac{1}{9}\{\tfrac{5}{9}u_2^3 - u_3^2 + (4u_2 u_3' + 5u_2' u_3) - 20(\tfrac{1}{2})^2(u_2 u_2'' + (u_2')^2) - 8(\tfrac{1}{2})^3 u_3''' + \tfrac{16}{3}(\tfrac{1}{2})^4 u_2''''\}$$

$$b_7 = \tfrac{1}{9}\{\tfrac{5}{3}u_2^2 u_3 + 5(u_3 u_3' - u_2^2 u_2') - \tfrac{20}{3}(\tfrac{1}{2})^2(5u_2'' u_3 + 7u_2' u_3' + u_2 u_3'')$$

$$-40(\tfrac{1}{2})^3(3u_2' u_2'' + u_2 u_2''') + \tfrac{16}{3}(\tfrac{1}{2})^4 u_3''''\}$$

$$b_8 = \tfrac{5}{27}(u_2 u_3^2 - \tfrac{2}{9}u_2^4) - \tfrac{5}{9}(u_2^2 u_3' - \tfrac{7}{3}u_2' u_2 u_3) + \tfrac{5}{81}(12u_3^{2'} + 31u_2 u_2^{2'}$$

$$+17u_2^2 u_2'' - 15u_3'' u_3) + \tfrac{5}{27}(10u_3'' u_2' + 13u_2'' u_3' + 7u_3 u_2''' + 3u_3'' u_2)$$

$$+\tfrac{5}{81}(8u_2^4 u_2 + 23u_2^{2''^2} + 32u_2' u_2''') + \tfrac{1}{81}u_2^{(6)}$$

$$(4.12)$$

Similarly, one can easily determine the coefficients a_{i+1} which are also expressed as functions of b_{i+1} and their derivatives. This result is summarized in the expression of $\mathcal{L}^{\frac{1}{3}}$

namely

$$\mathcal{L}^{\frac{1}{3}} = \partial \;+\; \tfrac{1}{3}u_2\partial^{-1}$$

$$+ \;\; \tfrac{1}{3}\{u_3 - \tfrac{1}{2}u_2'\}\partial^{-2}$$

$$- \;\; \tfrac{1}{9}\{u_2^2 + \tfrac{1}{2}^2 u_2''\}\partial^{-3}$$

$$+ \;\; \tfrac{1}{9}\{-2u_2u_3 + u_2'u_2 - \tfrac{1}{2}^2 u_3'' + \tfrac{1}{2}^3 u_2'''\}\partial^{-4}$$

$$+ \;\; \tfrac{1}{9}\{\tfrac{1}{3}\tfrac{1}{2}^4 u_2^{(4)} + u_2'u_3 - u_3^2 + \tfrac{5}{9}u_2^3\}.\partial^{-5}$$

$$+ \;\; \tfrac{1}{27}\{5u_2^2u_3 - 5\tfrac{1}{2}u_2^2u_2' + \tfrac{5}{2}(u_2'u_3' - u_2''u_3) + \tfrac{1}{2}^4 u_3^{(4)} - \tfrac{1}{2}^5 u_2^{(5)}\}.\partial^{-6}$$

$$+ \;\; \tfrac{1}{27}\{\tfrac{5}{9}u_2(9u_3^2 - 2u_2^3) - 5u_2'u_2u_3 + \tfrac{5}{3}\tfrac{1}{2}^2(6u_3'^2 - 6u_3''u_3 + 5u_2^2u_2'' - 2u_2u_2^{2'})$$

$$- \;\; 10\tfrac{1}{2}^3(-u_3''u_2' - u_3u_2''' + 2u_2''u_3') - \tfrac{10}{3}\tfrac{1}{2}^4(u_2^{(4)}u_2 + 4u_2'u_2''' - 5u_2''^2) - \tfrac{1}{3}\tfrac{1}{2}^6 u_2^{(6)}\}\partial^{-7}$$

$$+ \;\; ...$$

$$(4.13)$$

Furthermore, using the sl_3-Lax evolution equations

$$\frac{\partial\mathcal{L}}{\partial t_k} = [\mathcal{L}_+^{\frac{k}{3}}, \mathcal{L}], \qquad (4.14)$$

that we compute explicitly for $k = 1, 2, 4$ we obtain

$$\frac{\partial\mathcal{L}}{\partial t_1} \;=\; u_2'\partial + u_3' - \tfrac{1}{2}u_2''$$

$$\frac{\partial\mathcal{L}}{\partial t_2} \;=\; 2\{u_3' - \tfrac{1}{2}u_2''\}\partial - \tfrac{2}{3}\{u_2u_2' + (\tfrac{1}{2})^2 u_2'''\}$$

$$\frac{\partial\mathcal{L}}{\partial t_4} \;=\; \tfrac{4}{3}\{(u_2u_3)' - \tfrac{1}{2}(u_2''u_2 + u_2'^2) + 2(\tfrac{1}{2})^2 u_3''' - 2(\tfrac{1}{2})^3 u_2^{(4)}\}\partial \qquad (4.15)$$

$$+ \tfrac{4}{3}\{u_3u_3' - \tfrac{1}{3}u_2^2u_2' - \tfrac{1}{2}(u_2'u_3' + u_2''u_3) - (\tfrac{1}{2})^2(u_2'u_2'' + u_2u_2''') - \tfrac{2}{3}(\tfrac{1}{2})^4 u_2^{(5)}\}$$

Identifying both sides of the previous equations, one obtain the following first leading evolution equations

$$\frac{\partial}{\partial t_1} u_2 \quad = \quad u_2'$$

$$\frac{\partial}{\partial t_1} u_3 \quad = \quad u_3' - \frac{1}{2} u_2''$$

$$\frac{\partial}{\partial t_2} u_2 \quad = \quad 2u_3' - u_2''$$

$$\frac{\partial}{\partial t_2} u_3 \quad = \quad -\frac{2}{3} u_2 u_2' - \frac{2}{3}(\frac{1}{2})^2 u_2'''$$

$$\frac{\partial}{\partial t_4} u_2 \quad = \quad \frac{4}{3}\{(u_2 u_3)' - \frac{1}{2}(u_2'' u_2 + u_2'^2) + 2(\frac{1}{2})^2 u_3''' - 2(\frac{1}{2})^3 u_2^{(4)}\}$$

$$\frac{\partial}{\partial t_4}(u_3 - \frac{1}{2}u_2') = \frac{4}{3}\{u_3 u_3' - \frac{1}{3} u_2^2 u_2' - \frac{1}{2}(u_2' u_3' + u_2'' u_3) - (\frac{1}{2})^2(u_2' u_2'' + u_2 u_2''') - \frac{2}{3}(\frac{1}{2})^4 u_2^{(5)}.$$
(4.16)

These equations define what we call the sl_3-Boussinesq hierarchy. The first two equations are simply linear independent wave equations fixing the dimension of the first flow parameter t_1 to be $[t_1] = -1$.

The non trivial flow of this hierarchy starts really from the second couple of equations associated to t_2. It's important to point out that its important to deal with the basis of primary conformal fields v_k instead of the old basis u_k [10, 14], one can write the previous couple of equations in term of the spin 3 primary field $v_3 = u_3 - \frac{1}{2}u_2'$ as follows

$$\frac{\partial}{\partial t_2} u_2 \quad = \quad 2v_3'$$

$$\frac{\partial}{\partial t_2} v_3 \quad = \quad -\frac{2}{3}\{u_2 u_2' + (\frac{1}{2})^2 u_2'''\}$$
(4.17)

This couple of equations define the Boussinessq equation. Its second-order form is obtained by differentiating the first equation in Eq.(4.17) with respect to t_2 and then using the second equation. We find

$$\frac{\partial^2}{\partial t_2^2} u_2 = -\frac{4}{3}(u_2 u_2' + (\frac{1}{2})^2 u_2^{(3)})',$$
(4.18)

Recall that the classical Boussinesq equation is associated to the sl_3-Lax differential operator

$$\mathcal{L}_3 = \partial^3 + 2u\partial + v_3$$
(4.19)

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with $v_3 = u_3 - \frac{1}{2}u_2'$ defining the spin-3 primary field. This equation which takes the following form

$$u_{tt} = -(auu' + bu^{(3)})',\qquad(4.20)$$

where a, b are arbitrary constants, arises in several physical applications. Initially, it was derived to describe propagation of long waves in shallow water. This equation plays also a central role in $2d$ conformal field theories via its Gelfand-Dickey second Hamiltonian structure associated to the Zamolodchikov w_3 non linear algebra.

Similarly the third couple of equations Eq.(4.16) can be equivalently written as

$$
\begin{aligned}
\frac{\partial}{\partial t_4}u_2 &= \tfrac{4}{3}(u_2 v_3 + 2(\tfrac{1}{2})^2 v_3'')' \\[2mm]
\frac{\partial}{\partial t_4}v_3 &= \tfrac{4}{3}\{v_3 v_3' - (\tfrac{1}{2})^2 u_2 u_2''' - \tfrac{1}{3}(u_2^2 u_2' + 2(\tfrac{1}{2})^4 u_2^{(5)})\}
\end{aligned}
\qquad(4.21)
$$

To close work note that other flows equations associated to (sl_2)-KdV and (sl_3)-Boussinesq hierarchies can be also derived once some lengthly and hard computations are performed. One can also generalize the obtained results by considering other sl_n integrable hierarchies with $n > 3$.

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